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## Boson Condensation in an Einstein Universe

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**Abstract**

In this paper we investigate the Bose-Einstein condensation of massive spin-1 particles in an Einstein universe. The system is considered under relativistic conditions taking into consideration the possibility of particle-antiparticle pair production. An exact expression for the charge density is obtained, then certain approximations are employed in order to obtain the solutions in closed form. A discussion of the approximations employed in this and other work is given. The effects of finite-size and spin-curvature coupling are emphasized.

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# 1 Introduction

Since the early work of Altaie (1978), the study of Bose-Einstein condensation (BEC) in curved space has attracted the interest of many authors (Carvalho and Rosa 1980, Singh and Pathria 1984, Parker and Zhang 1991, Smith and Toms 1996, and Trucks 1998). This interest comes in the context of trying to understand the thermodynamics of the early universe and the role played by the finiteness of the space in determining the thermal behavior of bosons in the respective systems.

Most studies employ the static Einstein universe as the underlying geometry for the system, since the thermodynamic equilibrium in this universe can be defined without ambiguity (Altaie and Dowker 1978). In a time developing spacetime no such luxury is enjoyed, except for the Robertson-Walker spacetime which is conformally static (Kennedy 1978).

In an earlier work (Altaie 1978), we considered the cases of non-relativistic BEC of spin 0 and spin 1 bosons in an Einstein universe. We found that the finiteness of the system resulted in "smoothing out" the singularities of the thermodynamic functions found in the infinite systems, the enhancement of the condensate fraction and the displacement of the specific-heat maximum toward higher temperatures.

Carvalho and Rosa (1980) considered the case of relativistic scalar field in an Einstein universe in an effort to show that the finite size effects are negligible in comparison with the relativistic effects. However, a close look at their work shows that this comparison was done at implicitly large value of the radius  $a$  of the Einstein universe. And since finite size effects are proportional to  $1/a$  the finite size effects were found to be negligible. Moreover, the consideration of relativistic and ultra-relativistic BEC should take into account the possibility of particle-antiparticle production, since at temperatures greater than the rest mass of the particles, quantum field theory requires the inclusion of such a process (Harber and Weldon 1981, 1982).

Singh and Pathria (1984) considered the BEC of a relativistic conformally coupled scalar field in the Einstein universe and found qualitative and quantitative agreement with Altaie (1978). Parker and Zhang (1991) considered the ultra-relativistic BEC

of the minimally coupled scalar field in an Einstein universe in the limit of high temperatures. They showed, among other things that ultra-relativistic BEC can occur at very high temperature and densities in the Einstein universe, and by implication in the early stages of a dynamically changing universe. Parker and Zhang (1993) also showed that the Bose -Einstein condensate could act as a source for inflation leading to a de Sitter type universe.

Trucks (1998) repeated the calculations of Singh and Pathria but this time for the minimally coupled scalar field obtaining similar results. In fact performing the calculations for the minimally coupled scalar field amounts to substituting  $\overline{m} = (m^2 + 1/a^2)^{1/2}$  where  $m$  is the mass of the conformally coupled field. But as the calculations were effectively considered in the large radius region, the results come out to be identical with those of Singh and Pathria.

The importance of the study of BEC in curved spaces stems from the interest in understanding the thermodynamics of the very early universe and that such a phenomenon may shed some light on the problem of mass generation in the very early universe. Indeed the investigations of Toms (1992, 1993) have shown that a kind of symmetry breaking is possible.

In this paper we will consider the onset of BEC of the relativistic spin 1 particle-antiparticle system in an Einstein universe, a state which is surly relevant for the early stages of the universe at a point when the electromagnetic interactions decouple from the weak interactions. We will carry out the calculations in a similar fashion to that of Singh and Pathria and compare the results with our earlier non-relativistic case. Throughout this work we adopt the absolute system of units in which  $c = G = k = \hbar = 1$ .

## 2 The Charge Density

We consider an ideal relativistic Bose gas of spin 1 confined to the background geometry of the spatial section  $S^3$  of an Einstein universe with radius  $a$ . Since we are considering a relativistic system, it is necessary to take in consideration the possi-

bility of pair production. The system is taken to be formed of  $N_1$  particles and  $N_2$  antiparticles. The total number of particles  $Q = N_1 - N_2$  is assumed to be conserved, though  $N_1$  and  $N_2$  may change. A chemical potential  $\mu$  (is assigned for the particles and  $-\mu$  for the antiparticles (see Harber and Weldon 1981). Accordingly the particle and antiparticle distributions are given by,

$$N_1 = \sum_n d_n [e^{\beta(\epsilon_n - \mu)} - 1]^{-1}, \quad N_2 = \sum_n d_n [e^{\beta(\epsilon_n + \mu)} - 1]^{-1}, \quad (1)$$

where  $\beta = 1/T$ ,  $\epsilon_n$  are the eigen energies, and  $d_n$  is the degeneracy of the  $n$ th level. The equation of motion of the spin 1 field in an Einstein universe was considered by Schrödinger (1938) and the solution yields the following energy spectrum,

$$\epsilon_n = \frac{1}{a} (n^2 + m^2 a^2)^{1/2}, \quad (2)$$

with degeneracy,

$$d_n = 2(n^2 - 1), \quad n = 2, 3, 4, \dots \quad (3)$$

The charge density  $q$  is then found to be,

$$q = \frac{Q}{V} = \frac{4}{V} \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} (n^2 - 1) \sinh(l\beta\mu) \exp \left[ -l\beta' (m^2 a^2 + n^2)^{1/2} \right], \quad (4)$$

where  $V = 2\pi^2 a^3$  is the volume of the spatial section of the Einstein universe, and  $\beta' = 1/Ta$ . In order to carry out the summation over  $n$  in (4) we apply the Poisson summation formula (see Titchmarsh 1948),

$$\sum_{n=1}^{\infty} f(n) + \frac{1}{2}f(0) = \int_0^{\infty} f(t)dt + 2 \sum_{j=1}^{\infty} \int_0^{\infty} f(t) \cos(2\pi jt)dt. \quad (5)$$

Accordingly,

$$\sum_{n=1}^{\infty} (n^2 - 1) \exp \left[ -l\beta' (m^2 a^2 + n^2)^{1/2} \right] = \frac{1}{2} \exp(l\beta m) + I_0 + 2 \sum_j^{\infty} I_j, \quad (6)$$

where,

$$I_0 = \int_0^{\infty} (t^2 - 1) \exp \left[ -l\beta' (m^2 a^2 + t^2)^{1/2} \right] dt, \quad (7)$$

and

$$I_j = \int_0^{\infty} (t^2 - 1) \exp \left[ -l\beta' (m^2 a^2 + t^2)^{1/2} \right] \cos(2\pi jt)dt. \quad (8)$$

These integrals can be easily performed using (Gradshteyn and Ryzhik 1980)

$$\int_0^\infty \exp \left[ -\alpha \sqrt{\gamma^2 + x^2} \right] \cos(\lambda x) dx = \frac{\gamma \alpha}{\sqrt{\lambda^2 + \alpha^2}} K_1 \left( \gamma \sqrt{\lambda^2 + \alpha^2} \right), \quad (9)$$

where  $K_\nu$  are the modified Bessel functions of the second kind.

Evaluating  $I_0$  and  $I_j$  and then substituting in (4) we find that the charge density can be written as,

$$\begin{aligned} q = & \frac{1}{2\pi^2 a^3} \left[ \frac{1}{e^{\beta(m-\mu)} - 1} - \frac{1}{e^{\beta(m+\mu)} - 1} \right] \\ & - \frac{2m}{\pi^2 a^2} \sum_{l=1}^\infty \sum_{j=1}^\infty l \sinh(l\beta\mu) \left[ \frac{K_1(l\beta m)}{l} + \frac{K_1(\beta m z)}{z} \right] \\ & + \frac{2m^2}{\pi^2 \beta} \sum_{l=1}^\infty \sum_{j=-\infty}^\infty l \sinh(l\beta\mu) \left[ \frac{K_2(\beta m z)}{z^2} - (\beta m j')^2 \frac{K_3(\beta m z)}{z^3} \right], \end{aligned} \quad (10)$$

where

$$z = \sqrt{l^2 + j'^2} \quad \text{and} \quad j' = (2\pi a/\beta)j. \quad (11)$$

The first term in (10) arises because the  $n = 0$  term in the summation over  $n$  in (4) is non-zero. The second term arises because of the spin-curvature coupling, which is defined mathematically by the coefficient  $a_n$  of the Schwinger-De Witt expansion (see, De Witt 1965). The third term is just twice that of the scalar case considered by Singh and Pathria (1984). Both the first and the second terms disappear in the limit  $a \rightarrow \infty$ . The bulk term is obtained in this limit assuming that  $q$  remains constant, which is the known thermodynamic limit. This reduces to the  $j = 0$  contribution of the last term in (10) which gives,

$$q_B(\beta, \mu) = \frac{2m^3}{\pi^2} \sum_{l=1}^\infty (l\beta m)^{-1} \sinh(l\beta\mu) K_2(l\beta m). \quad (12)$$

This is just twice the value obtained for the scalar case as would be expected. The summation over  $j$  in (10) can be performed using the Poisson summation formula (5) again, where we obtain,

$$\begin{aligned} q = & q_B(\beta, \mu) + \left[ \frac{1}{e^{\beta(m-\mu)} - 1} - \frac{1}{e^{\beta(m+\mu)} - 1} \right] \\ & - \frac{2m}{\pi^2 a^2} \sum_{l=1}^\infty \sum_{p=1}^\infty \int_0^\infty l \sinh(l\beta\mu') \left[ \frac{K_1(l\beta m)}{l} + \frac{K_1(\beta m z)}{z} \right] dj \end{aligned}$$

$$+\frac{4m^2}{\pi^2\beta}\sum_{l=1}^{\infty}\sum_{p=1}^{\infty}\int_0^{\infty}l\sinh(l\beta\mu')\left[\frac{K_2(\beta mz)}{z^2}-(\beta mj)^2\frac{K_3(\beta mz)}{z^3}\right]dj, \quad (13)$$

where  $\mu' = \mu + 2\pi ip$ . The integrals in the above equation can be evaluated exactly using (see Singh and Pathria 1984),

$$\int_0^{\infty}j\sinh(jx)\frac{K_{\nu}\left[y(j^2+\xi^2)^{1/2}\right]}{(j^2+\xi^2)^{\nu/2}}dj=\left(\frac{\pi\xi^3}{2}\right)^{1/2}\frac{x(y^2-x^2)^{\nu/2-3/4}}{(\xi y)^{\nu}}K_{\nu-3/2}\left[\xi(y^2-x^2)^{1/2}\right]. \quad (14)$$

Then using (see Gradshteyn and Ryzhik 1980),

$$\int_0^{\infty}\sinh(ax)K_1(bx)dx=\frac{\pi}{2}\frac{a}{b\sqrt{b^2-a^2}}, \quad (15)$$

and the relation,

$$K_{1/2}(z)-zK_{3/2}(z)=-zK_{-1/2}(z)=-\left(\frac{\pi z}{2}\right)^{1/2}e^{-z}, \quad (16)$$

we obtain

$$q=q_B(\beta,\mu)+\frac{1}{2\pi^2a^3}\left[\frac{1}{e^{\beta(m-\mu)}-1}-\frac{1}{e^{\beta(m+\mu)}-1}\right]-\frac{2}{\pi\beta}\sum_{l=1}^{\infty}\sum_{p=-\infty}^{\infty}\mu'\left[\frac{1}{2a^2\tau}+\left(\frac{1}{a^2\tau}+\tau\right)e^{-2\pi al\tau}\right], \quad (17)$$

where  $\tau=\sqrt{m^2-\mu'^2}$ .

The summation over  $l$  can be easily done, so the exact expression for the charge density becomes,

$$q=q_B(\beta,\mu)+\frac{1}{2\pi^2a^3}\left[\frac{1}{e^{\beta(m-\mu)}-1}-\frac{1}{e^{\beta(m+\mu)}-1}\right]-\frac{2}{\pi\beta}\sum_{p=-\infty}^{\infty}\mu'\left[\frac{1}{2a^2\tau}+\left(\frac{1}{a^2\tau}+\tau\right)\frac{1}{e^{2\pi a\tau}-1}\right]. \quad (18)$$

Note that this form of the charge density is exact and no approximation whatsoever has been made through the calculation.

### 3 Bose-Einstein Condensation

We will adopt the microscopic criteria for marking the onset of the condensation (Altaie 1978), according to which the condensation region is defined such that a

large number of particles is found occupying the ground state. This implies that the chemical potential  $\mu$  of the system approaches the minimum single particle energy  $\epsilon_2$ , not the single rest mass energy considered by Singh and Pathria (1984). This can be observed directly from (1). However the consideration of the criteria that  $\mu \rightarrow m$  is justified only for the minimally coupled scalar field case where the minimum energy is  $m$  (see Parker and Zhang 1991). In our case the chemical potential must satisfy the condition that

$$-\left(m^2 + \frac{4}{a^2}\right)^{1/2} < \mu < \left(m^2 + \frac{4}{a^2}\right)^{1/2}, \quad (19)$$

whereas in the conformally coupled spin 0 case the condition is,

$$-\left(m^2 + \frac{1}{a^2}\right)^{1/2} < \mu < \left(m^2 + \frac{1}{a^2}\right)^{1/2}. \quad (20)$$

However, if  $m^2$  is much larger than  $1/a^2$  then it will be justified to take the limit  $\mu \rightarrow m$ , but this approximation may impose certain restrictions on the range of the region under consideration. We will follow Singh and Pathria and adopt such approximation in this work. In such a case the main contribution of the summation over  $p$  in (18) comes from the  $p = 0$  term. Other terms are of order of  $e^{-a\sqrt{m/\beta}}$ , i.e.  $O(e^{-a/\lambda_T})$  where  $\lambda_T = \sqrt{2\pi\beta/m}$  is the mean thermal wavelength of the particle. The bulk term will reduce to (Singh and Pandita 1983),

$$q_B(\beta, \mu) = q_B(\beta, m) - \frac{m}{\pi\beta} (m^2 - \mu^2)^{1/2} + O(m^2 - \mu^2). \quad (21)$$

Therefore we can write

$$\begin{aligned} q_B(\beta, \mu) &\approx q_B(\beta, m) - \frac{m}{\pi\beta} (m^2 - \mu^2)^{1/2} \\ &- \frac{2\mu}{\pi\beta} \left[ (m^2 - \mu^2)^{1/2} - \frac{1}{a^2} (m^2 - \mu^2)^{-1/2} \right] \frac{1}{\exp(2\pi a \sqrt{m^2 - \mu^2}) - 1} \\ &+ \frac{1}{2\pi^2 a^3} \left[ \frac{1}{e^{\beta(m-\mu)} - 1} - \frac{1}{e^{\beta(m+\mu)} - 1} \right]. \end{aligned} \quad (22)$$

If we define the thermogeometric parameter  $y$  as

$$y = \pi a (m^2 - \mu^2)^{1/2}, \quad (23)$$

then equation (22) becomes

$$q \approx q_B(\beta, m) - \frac{m}{\pi\beta a} \left[ \left( \frac{y}{\pi} + \frac{\pi}{y} \right) \coth y - \frac{\pi}{y^2} \right]. \quad (24)$$

From this equation the behavior of the thermogeometric parameter  $y$  in the condensation region can be determined. It is clear that the second term in (24) defines the finite-size and spin-curvature effects.

## 4 The Condensate Fraction

The growth of the condensate fraction is studied here in comparison with the bulk case. This will show the finite-size and the spin-curvature effect in the range considered, i.e.,  $ma \gg 1$ . The charge density in the ground state is obtained if we substitute  $n = 2$  in (1). This gives

$$q_0 = \frac{6}{2\pi^2 a^3} \left[ \left( e^{\beta(\epsilon_2 - \mu)} - 1 \right)^{-1} - \left( e^{\beta(\epsilon_2 + \mu)} - 1 \right)^{-1} \right]. \quad (25)$$

In the condensation region  $\mu \rightarrow \epsilon_2$ , so that the main contribution to the charge density in the ground state comes from the first term in the square bracket. This means that

$$q_0 \approx \frac{3}{\pi^2 a^3 \beta (\epsilon_2 - \mu)}. \quad (26)$$

From (23) as  $\mu \rightarrow m$ , we have

$$\mu \approx m \left( 1 - \frac{y^2}{2\pi^2 a^2 m^2} \right). \quad (27)$$

On the same footing we can expand  $\epsilon_2$  as

$$\epsilon_2 \approx m \left( 1 + \frac{2}{m^2 a^2} \right). \quad (28)$$

So that (26) becomes

$$q_0 \approx \frac{6m}{a\beta(y^2 + 4\pi^2)}. \quad (29)$$

This means that the macroscopic growth of the condensate will occur only when  $y^2 \rightarrow -4\pi^2$ , (i.e.,  $y \rightarrow 2\pi i$ ). In order to see how this condensate compares with the bulk case we use the expansion:

$$\coth y = \frac{1}{y} + 2y \sum_{k=1}^{\infty} \frac{1}{y^2 + \pi^2 k^2}. \quad (30)$$



So that,

$$\left(\frac{y}{\pi} + \frac{\pi}{y}\right) \coth y - \frac{\pi}{y^2} = \frac{5}{\pi} - \frac{6\pi}{y^2 + 4\pi^2} + \frac{2}{\pi}(y^2 + \pi^2) \sum_{k=3}^{\infty} \frac{1}{y^2 + \pi^2 k^2}. \quad (31)$$

Substituting this in (24) and using (29) we get,

$$q_0 = q - q_B(\beta, m) + \frac{2m}{\pi^2 \beta a} \left[ \frac{5}{2} + (y^2 + \pi^2) \sum_{k=3}^{\infty} \frac{1}{y^2 + \pi^2 k^2} \right]. \quad (32)$$

For the bulk system ( $a \rightarrow \infty$ ) there exists a critical temperature,  $T = T_c$  given by

$$q_B(\beta_c, m) = q. \quad (33)$$

This condition can be written as

$$q = q_B(\beta_c, m) = \frac{m^3}{\pi^2} \sum_{l=1}^{\infty} (l\beta_c m)^{-1} \sinh(l\beta_c m) K_2(l\beta_c m) = \frac{m^3}{2\pi^2} W(\beta_c, m), \quad (34)$$

where

$$W(\beta, \mu) = 2 \sum_{l=1}^{\infty} (l\beta\mu)^{-1} \sinh(l\beta\mu) K_2(l\beta\mu). \quad (35)$$

Thus we can write the bulk condensate density as

$$(q_0)_B = \begin{cases} 0 & \text{for } T > T_c \\ q \left(1 - \frac{W(\beta, m)}{W(\beta_c, m)}\right) & \text{for } T < T_c \end{cases}. \quad (36)$$

For the case of the finite system under consideration the condensate density is given by

$$q_0 = q \left[ 1 - \frac{W(\beta, m)}{W(\beta_c, m)} \right] + \frac{2m}{\pi^2 \beta a} \left[ \frac{5}{2} + (y^2 + \pi^2) \sum_{k=3}^{\infty} \frac{1}{y^2 + \pi^2 k^2} \right], \quad (37)$$

where the  $y$  dependence on  $T$  in the condensation region can now be determined explicitly by

$$\left(\frac{y}{\pi} + \frac{\pi}{y}\right) \coth y - \frac{\pi}{y^2} = -\frac{\pi \beta a q}{m} \left[ 1 - \frac{W(\beta, m)}{W(\beta_c, m)} \right]. \quad (38)$$

This equation can be solved numerically for the values of  $y$  at different temperatures. However, we notice that at  $T = T_c$  ( $W(\beta, m) = W(\beta_c, m)$ ), the value of  $y$  can be obtained by solving the equation,

$$\left(\frac{y_c}{\pi} + \frac{\pi}{y_c}\right) \coth y_c = \frac{\pi}{y_c^2}, \quad (39)$$

which has a solution at  $y_c = 4.859i$ .

## 5 Non-relativistic and Ultra-relativistic Limits

The non-relativistic limit is obtained by setting  $\beta m \gg 1$ . In this case we can use the asymptotic expansion of the Bessel functions of the second kind for large argument, where we have (see Abramowitz and Stegun 1968),

$$K_2(j\beta m) \approx \left(\frac{\pi}{2j\beta m}\right)^{1/2} e^{-j\beta m} \left[1 + \frac{15}{8} \frac{1}{j\beta m} + \frac{105}{128} \frac{1}{(j\beta m)^2} + \dots\right]. \quad (40)$$

So that

$$[W(\beta, m)]_{NR} = \left(\frac{\pi}{2m^3}\right)^{1/2} \zeta(3/2) \beta^{-3/2} \quad (41)$$

and

$$\left[\frac{W(\beta, m)}{W(\beta_c, m)}\right]_{NR} = \left(\frac{\beta_c}{\beta}\right)^{3/2} = \left(\frac{T}{T_c}\right)^{3/2}. \quad (42)$$

Therefore from (36) we have for the bulk system,

$$(q_0)_B = q \left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right], \quad T < T_c \quad (43)$$

In the non-relativistic limit (38) reduces to

$$2 \left(\frac{y}{\pi} + \frac{\pi}{y}\right) \coth y - \frac{2\pi}{y^2} = x^{1/2} \left[1 - x^{-3/2}\right] \frac{a}{\bar{l}} [2\zeta(3/2)]^{2/3}, \quad (44)$$

where  $x = T/T_c$  and  $\bar{l}$  is the mean inter-particle distance. Note that equation (44) is the corrected version of equation (62) of our earlier paper (Altaie 1978).

In the ultra-relativistic limit  $\beta m \ll 1$ , therefore, we use the expansion,

$$K_2(j\beta m) \sim \frac{1}{2} \Gamma(2) \left(\frac{1}{2} j\beta m\right)^{-2}. \quad (45)$$

So that,

$$[W(\beta, m)]_{UR} \sim \frac{2\pi^2}{3m^2} \beta^{-2}. \quad (46)$$

Accordingly the ultra-relativistic behavior of the bulk charge density given in (36) will be

$$(q_0)_B = q \left[1 - \frac{T^2}{T_c^2}\right]. \quad (47)$$

However, substituting (46) in (34), the critical temperature for the bulk spin 1 particles is

$$T_c = \sqrt{\frac{3q}{2m}}. \quad (48)$$

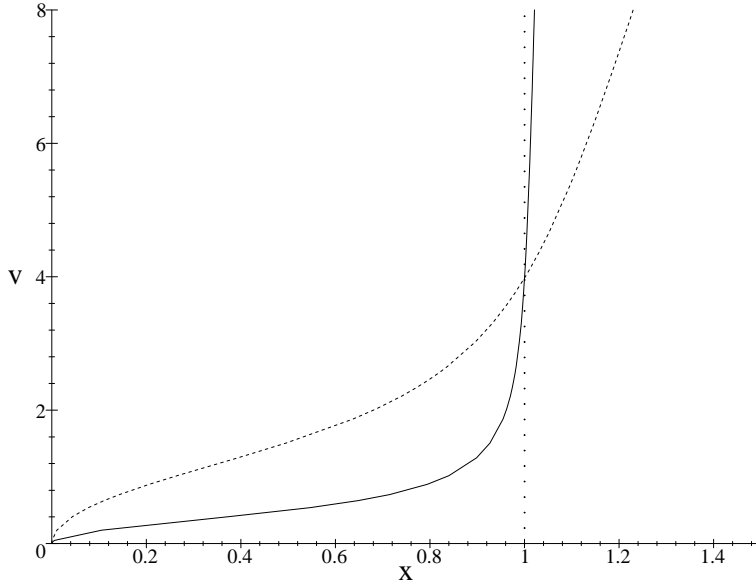


Figure 1: The quantity  $v = \sqrt{y^2 + 4\pi^2}$ , where  $y$  is the thermogeometric parameter, is plotted as a function of the scaled temperature  $x = T/T_c$  for two different values of the radius  $a$ . The solid line is for the larger value of  $a$ . The vertical dotted line at  $x = 1$  is marked for reference.

This is the analogous result to that of the minimally coupled scalar field. This shows that the treatment of the problem using the large radius approximation as adopted by Singh and Pathria and in this work is equivalent to the high temperature expansion of Parker and Zhang (1991). This can be understood in the light of the fact that in the ultra-relativistic regime  $T \propto a^{-1}$ , so that  $aT = \text{constant}$ . In the ultra-relativistic regime the thermogeometric parameter  $y$  behaves according to

$$\left(\frac{y}{\pi} + \frac{\pi}{y}\right) \coth y = \frac{\pi a q}{m T_c} \left(x - \frac{1}{x}\right) = \pi a \sqrt{\frac{2q}{3m}} \left(x - \frac{1}{x}\right). \quad (49)$$

This equation can be solved for given values of  $a$ ,  $q$ , and  $m$ . For illustrative purposes, figure 1 shows the behavior of the thermogeometric parameter  $y$  (drawn as  $v = \sqrt{y^2 + 4\pi^2}$ ) versus the scale temperature  $x = T/T_c$ . It is clear that as  $a$  increases the

quantity  $\sqrt{y^2 + 4\pi^2}$  tends to a step function.

## 6 The Critical Radius

Although it is known that the universe as a whole is neutral or almost neutral, the observational limit on the net average charge density do not exclude the possibility that the charge density in the early universe was sufficient to produce relativistic Bose-Einstein condensation. Following the assumption that the charge is conserved we can write

$$q_i = \left(\frac{a_p}{a_i}\right)^3 q_p, \quad (50)$$

where  $q_i$  and  $q_p$  are the initial and the present charge densities, respectively. If we consider  $Ta = C$  and assume that  $C$  remains constant throughout the development of the universe then from (48) we can allocate a critical radius for the universe  $a_c$  below which the gas will be always in the condensate state. Parker and Zhang (1991) have already noted this. However for the spin 1 case this critical radius will be given by

$$a_c = \frac{3q_p a_p^3}{2mC^2}. \quad (51)$$

The estimated upper bound on the net average charge density of the universe at present is  $q < 10^{-24} \text{ cm}^{-3}$  (see Dolgov and Zeldovich 1981). If this upper bound on the present charge density is adopted, then an upper bound on the radius of the universe at which condensate starts can be deduced. Using a value of  $10^{28}$  for  $C$  (Trucks 1998) and  $a_p = 10^{28} \text{ cm}$ , the critical radius for the onset of the condensation of the heavy gauge bosons  $W$  can be calculated. This gives  $a_c < 10^{-11} \text{ cm}$ .

## 7 Discussion

In the previous sections we have investigated the behavior of an ideal relativistic spin-1 gas confined to the background geometry of an Einstein universe. The extension of the problem into the Robertson-Walker spacetime is straight forward and results will be similar to the ones obtained here apart from the fact that in the Robertson-Walker

spacetime the radius  $a$  is time-dependent, so  $a \rightarrow a(t)$ . On the other hand more accurate calculations may be needed in order to understand the behavior of bosons at very early stages of the development of the universe. In this context the problem of photon condensation will surely be of great interest, however some technical and conceptual problems hinders obtaining exact results.

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